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Structural elucidation of the mean square of the Hurwitz zeta-function

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Abstract

In this note we shall give a complete structural description of the mean square of the Hurwitz zeta-function whose study was started 50 years ago. Instead of appealing to Atkinson's dissection, we incorporate the built-in structure of the Hurwitz zeta-function as the solution of the difference equation. First we shall prove a Katsurada–Matsumoto formula from which the best asymptotic expansion for the mean square at $\frac{1}{2}+it$ follows by K-times integration by parts, and then we shall show that their fundamental formula is essentially the N-times integration by parts of the same formula. The key is to introduce a suitable function f_κ the integration of which gives $\int_1^2 \zeta(u,x)\zeta(v,x)\,\mathrm{d}x$, and then to view $\int_1^2 f_\kappa(x;u,v)\,\mathrm{d}x$ as $\int_1^\infty -\Delta f_\kappa(x;u,v)\,\mathrm{d}x$.

1. Introduction and deduction of the basic formula

In this note we shall give a complete description of the structure of the 50 years old problem [15] on the asymptotic behavior of the mean square

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$$H(s)=\int_1^2 \left|\zeta(s,x)\right|^2 \mathrm{d}x \tag{1.1}$$

of the Hurwitz zeta-function defined by

$$\zeta(s,x)=\sum_{n=0}^{\infty}(n+x)^{-s}, \tag{1.2}$$

$\sigma=\operatorname{Re}s>1$, $x>0$ and then continued analytically over the whole plane with a simple pole at $s=1$.

This problem (we refer the reader to the excellent survey of Matsumoto [17]) was first studied by Koksma and Lekkerkerker [15] whose result was used by Gallagher [8] on his result on the mean square of Dirichlet L-functions. Then Balasubramanian [4], Rane [19], Sitaramachandrarao [20], Zhang [25,26] obtained asymptotic formula for (1.1) with $s=\frac{1}{2}+it$. Then Anderson [1] and Zhang [27] obtained a rather precise asymptotic formula. Finally, Katsurada and Matsumoto [12,13] obtained the asymptotic expansion for $H(\frac{1}{2}+it)$. Their main idea was to use Atkinson's dissection method [2,3] adapted by Motohashi [18] in his research on Dirichlet L-functions.

It is Katsurada [11] who gave a clearer description of the problem from the point of view of the Mellin–Barnes integral and hypergeometric functions. There is, however, one black box in Katsurada's proof, i.e. Lemma 3 which gives

$$\frac{1}{w-z}=\frac{1}{2\pi i}\int_{(\rho_0)}\frac{\Gamma(w)\Gamma(z+r)\Gamma(1+r)\Gamma(-r)}{\Gamma(z)\Gamma(w+1+r)}e^{\pi ir}\,\mathrm{d}r,$$

which he had to substitute for $\frac{1}{u+v+s-1}$ in the integral of (4.8) [11], which is hard to be called transparent, nor the use of Mellin–Barnes integrals.

Our approach appeals neither to Atkinson's dissection nor to Mellin–Barnes integrals, but to the structure of the Hurwitz zeta-function as a principal solution of the difference equation, which is the point stated, e.g., in Erdélyi [7, (2), p. 24]. The idea of incorporating this structure of the Hurwitz zeta-function arose in our course of researches on the product of zeta-functions. Here we use the telescoping series technique to view the integral $\int_1^2 f_\kappa(x;y,v)\,\mathrm{d}x$ of a certain function, whose definition will be seen from the convergence conditions in Section 2, as $\int_1^\infty -\Delta f_\kappa(x;u,v)\,\mathrm{d}x$, with Δ denoting the difference operator acting on x:

$$\Delta f_\kappa(x;u,v)=f_\kappa(x+1;u,v)-f_\kappa(x;u,v). \tag{1.3}$$

In Section 1, we shall exhibit the case $\kappa=0$, without paying much attention to the convergence problem, in order to show the clearer picture of the situation and also the relevance to Wilton's earlier work [23,24].

Thus, for $u\neq 1$, $v\neq 1$, $u+v\neq 1$, we introduce

$$f_0(x;u,v)=\zeta(u,x)\zeta(v,x)-\zeta(u+v,x)-\zeta(u+v-1,x)\left(\frac{1}{u-1}+\frac{1}{v-1}\right), \tag{1.4}$$

by which we are naturally led to remember Wilton's work [23,24] and subsequent work of Bellman [5]. $f_0(x;u,v)$ is the incomplete gamma series in their research, which are hardly tractable.

Cf. in this regard, Lavrik's [16] clear description of various incomplete gamma series expression for zeta- and L-functions.

Let Δ denote the difference operator defined by (1.3). Then

$$-\Delta f_0(x;u,v) = g_0(x;u,v) + g_0(x;v,u), \tag{1.5}$$

where

$$g_0(x;u,v) = x^{-v}\left(\zeta(u,x+1) - \frac{1}{u-1}x^{1-u}\right), \tag{1.6}$$

and $g_0(x;v,u)$ with u, v changed in (1.6).

Now, by (1.4), and the known result that

$$\int_1^2 \zeta(s,x)\,\mathrm{d}x = \frac{1}{s-1}, \quad s \neq 1, \tag{1.7}$$

we deduce that

$$\int_1^2 f_0(x;u,v)\,\mathrm{d}x = \int_1^2 \zeta(u,x)\zeta(v,x)\,\mathrm{d}x - \frac{1}{u+v-1} - \frac{1}{u+v-2}\left(\frac{1}{u-1} + \frac{1}{v-1}\right),$$

for $u+v \neq 1, 2$, $u \neq 1$, $v \neq 1$, or

$$\int_1^2 \zeta(u,x)\zeta(v,x)\,\mathrm{d}x = \int_1^2 f_0(x;u,v)\,\mathrm{d}x + \frac{1}{u+v-1} + \frac{1}{u+v-2}\left(\frac{1}{u-1} + \frac{1}{v-1}\right). \tag{1.8}$$

We want to apply the telescoping series technique to $\int_1^2 f_0(x;u,v)\,\mathrm{d}x$ to think of it as $\int_1^\infty -\Delta f_0(x;u,v)\,\mathrm{d}x$. It is at this point that we take the convergence into account. Without any aid, this infinite integral is convergent for $\operatorname{Re} u + \operatorname{Re} v > 1$ only, which we suppose for the present.

Hence we have, by (1.5),

$$\int_1^2 f_0(x;u,v)\,\mathrm{d}x = \int_1^\infty -\Delta f_0(x;u,v)\,\mathrm{d}x = \int_1^\infty g_0(x;u,v)\,\mathrm{d}x + \int_1^\infty g_0(x;u,v)\,\mathrm{d}x. \tag{1.9}$$

It is enough to consider the first integral on the right of (1.9). Integrating by parts, we deduce that

$$\int_1^\infty g_0(x;u,v)\,\mathrm{d}x = -S_1(u,v) - \frac{1}{u-1}\frac{1}{v-1} + \frac{1}{1-v}\int_1^\infty x^{1-v}\big(u\zeta(u+1,x+1) - x^{-u}\big)\,\mathrm{d}x, \tag{1.10}$$

where as in [13],

$$S_N(u,v)=\sum_{n=0}^{N-1}\frac{(u)_n}{(1-v)_{n+1}}\bigl(\zeta(u+n)-1\bigr), \tag{1.11}$$

and $(s)_n=\Gamma(s+n)/\Gamma(s)$ signifying the Pochhammer symbol.

We want to complete the integral over $(1,\infty)$ by adding the corresponding one over $(0,1)$. Since we have

$$\frac{u}{1-v}\int_0^1 x^{1-v}\zeta(u+1,x+1)\,\mathrm{d}x=T_1(u,v), \tag{1.12}$$

where as in [14]

$$T_N(u,v)=\frac{(u)_N}{(1-v)_N}\sum_{l=1}^{\infty}l^{1-u-v}\int_l^{\infty}\beta^{u+v-2}(1+\beta)^{-u-N}\,\mathrm{d}\beta, \tag{1.13}$$

and for $\operatorname{Re}u+\operatorname{Re}v<2$,

$$-\frac{1}{1-v}\int_0^1 x^{1-v}x^{-u}\,\mathrm{d}x=\frac{1}{1-v}\frac{1}{2-u-v}=-\frac{1}{u-1}\left(\frac{1}{v-1}-\frac{1}{u+v-2}\right), \tag{1.14}$$

it follows from (1.10) by adding

$$\frac{1}{1-v}\int_0^{\infty}x^{1-v}\bigl(u\zeta(u+1,x+1)-x^{-u}\bigr)\,\mathrm{d}x$$

and subtracting (1.12) and (1.14) that

$$\begin{aligned}\int_1^{\infty}g_0(x;u,v)\,\mathrm{d}x&=-S_1(u,v)-T_1(u,v)+\frac{1}{1-v}\int_0^{\infty}x^{1-v}\bigl(u\zeta(u+1,x+1)-x^{-u}\bigr)\,\mathrm{d}x\\&\quad-\frac{1}{u-1}\frac{1}{u+v-2}.\end{aligned} \tag{1.15}$$

Hence, substituting (1.15) and its counterpart for $g_0(x;u,v)$ in (1.9), we conclude that

$$\begin{aligned}&\int_1^2 f_0(x;u,v)\,\mathrm{d}x\\&\quad=-\frac{1}{u-1}\frac{1}{u+v-2}-\frac{1}{v-1}\frac{1}{u+v-2}+\frac{u}{1-v}\int_0^{\infty}x^{1-v}\left(\zeta(u+1,x+1)-\frac{1}{u}x^{-u}\right)\mathrm{d}x\end{aligned}$$

$$+\frac{v}{1-u}\int_0^\infty x^{1-u}\left(\zeta(v+1,x+1)-\frac{1}{v}x^{-v}\right)\mathrm{d}x$$
$$-S_1(u,v)-T_1(u,v)-S_1(v,u)-T_1(v,u). \tag{1.16}$$

The final step is to express the above Mellin transforms in closed form. This is readily done by using the well-known expression ($\operatorname{Re} u+\operatorname{Re} v>1$)

$$\zeta(u+1,x+1)=\frac{1}{\Gamma(u+1)}\int_0^\infty \frac{t}{e^t-1}e^{-xt}t^{u-1}\,\mathrm{d}t \tag{1.17}$$

and

$$\frac{1}{u}x^{-u}=\frac{1}{\Gamma(u+1)}\int_0^\infty e^{-xt}t^{u-1}\,\mathrm{d}t. \tag{1.18}$$

Substituting (1.17) and (1.18) for the integrand of the integral on the right of (1.16), and changing the order of integration, we deduce that

$$\begin{aligned}
&\frac{u}{1-v}\int_0^\infty x^{1-v}\left(\zeta(u+1,x+1)-\frac{1}{u}x^{-u}\right)\mathrm{d}x\\
&\quad=\frac{1}{(1-v)\Gamma(u)}\int_0^\infty x^{1-v}\int_0^\infty\left(\frac{t}{e^t-1}-1\right)e^{-xt}t^{u-1}\,\mathrm{d}t\,\mathrm{d}x\\
&\quad=\frac{1}{(1-v)\Gamma(u)}\int_0^\infty\left(\frac{t}{e^t-1}-1\right)t^{u-1}\int_0^\infty x^{1-v}e^{-xt}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\frac{\Gamma(2-v)}{(1-v)\Gamma(u)}\int_0^\infty\left(\frac{t}{e^t-1}-1\right)t^{u+v-3}\,\mathrm{d}t
\end{aligned}$$

by the change of variable. The last Mellin transform is another well-known formula for the Riemann zeta-function:

$$\Gamma(u+v-1)\zeta(u+v-1).$$

Thus, (1.16) can be reduced to

$$\begin{aligned}
\int_1^2 f_0(x;u,v)\,\mathrm{d}x&=-\frac{1}{u-1}\frac{1}{u+v-2}-\frac{1}{v-1}\frac{1}{u+v-2}\\
&\quad+\left(\frac{\Gamma(1-v)}{\Gamma(u)}+\frac{\Gamma(1-u)}{\Gamma(v)}\right)\Gamma(u+v-1)\zeta(u+v-1)\\
&\quad-S_1(u,v)-S_1(v,u)-T_1(u,v)-T_1(v,u).
\end{aligned} \tag{1.19}$$

Substituting (1.19) in (1.8), we finally obtain

$$\int_1^2 \zeta(u,x)\zeta(v,x)\,\mathrm{d}x = \frac{1}{u+v-1} + \Gamma(u+v-1)\zeta(u+v-1)\left(\frac{\Gamma(1-v)}{\Gamma(u)} + \frac{\Gamma(1-u)}{\Gamma(v)}\right)$$
$$- S_1(u,v) - S_1(v,u) - T_1(u,v) - T_1(v,u), \tag{1.20}$$

which is the case $N=1$ of [17, (11.4)]. As is remarked there, the best asymptotic expansion [17, (11.2)] follows from (1.20) by integrating $T_1(u,v)$ and $T_1(v,u)$ K-times by parts, and letting $u \to \frac{1}{2}+it$, $v \to \frac{1}{2}+it$.

Theorem. [14] *For any integer $N \geqslant 1$, suppose two independent complex variables u and v lie in the range $-N+1 < \operatorname{Re} u$, $\operatorname{Re} v < N+1$, $\operatorname{Re} u + \operatorname{Re} v \notin \mathbb{Z}$ and $(u,v) \notin E$, the set of all possible singularities of the terms appearing on the right side of* (1.21). *Then*

$$\int_1^2 \zeta(u,x)\zeta(v,x)\,\mathrm{d}x = \frac{1}{u+v-1} + \Gamma(u+v-1)\zeta(u+v-1)\left(\frac{\Gamma(1-v)}{\Gamma(u)} + \frac{\Gamma(1-u)}{\Gamma(v)}\right)$$
$$- S_N(u,v) - S_N(v,u) - T_N(u,v) - T_N(v,u), \tag{1.21}$$

where $S_N(u,v)$ and $T_N(u,v)$ are defined in (1.11) *and* (1.13), *respectively.*

Thus we may say that the known asymptotic expansions of $H(s)$ can be obtained by integration by parts, save for the convergence problem.

2. Widening the region of convergence and the case of the Hurwitz–Lerch zeta-function

First we shall clarify why we are led to the definition of $f_0(x;u,v)$ in (1.4). This is visible when we accelerate the convergence of the infinite integral

$$\int_1^\infty -\Delta f_0(x;u,v)\,\mathrm{d}x.$$

Recall the well-known expansion in the second variable of the Hurwitz zeta-function [9,21],

$$\zeta(s,x+1) = \sum_{n=0}^{\kappa} \frac{\Gamma(s+n-1)}{\Gamma(s)n!} B_n x^{-s-n+1} + O\left(x^{-\sigma-\kappa}\right) \tag{2.1}$$

for any non-negative integer κ, where B_n stands for the nth Bernoulli number. Hence, if we take κ so large, that

$$\kappa > 1 - \operatorname{Re} u - \operatorname{Re} v, \tag{2.2}$$

then in the region

$$\operatorname{Re} u + \operatorname{Re} v > 1 - \kappa, \tag{2.3}$$

we have

$$x^{-v}\left(\zeta(u,x+1)-\sum_{n=0}^{\kappa}\frac{\Gamma(u+n-1)}{\Gamma(u)n!}B_n x^{-u-n+1}\right)=O\left(x^{-\operatorname{Re}u-\operatorname{Re}v-\kappa}\right). \tag{2.4}$$

Therefore, if we can define $f_\kappa(x;u,v)$ in such a way that

$$\begin{aligned}-\Delta f_\kappa(x;u,v)&=x^{-v}\left(\zeta(u,x+1)-\sum_{n=0}^{\kappa}\frac{\Gamma(u+n-1)}{\Gamma(u)n!}B_n x^{-u-n+1}\right)\\&\quad+x^{-u}\left(\zeta(v,x+1)-\sum_{n=0}^{\kappa}\frac{\Gamma(v+n-1)}{\Gamma(v)n!}B_n x^{-v-n+1}\right),\end{aligned} \tag{2.5}$$

then

$$\Delta f_\kappa(x;u,v)=O\left(x^{-\operatorname{Re}u-\operatorname{Re}v-\kappa}\right),$$

so that in the region (2.3) we surely have

$$\int_1^\infty \left|-\Delta f_\kappa(x;u,v)\right|\,\mathrm{d}x<\infty. \tag{2.6}$$

Recalling from our another work [10] on the product of zeta-functions that

$$-\Delta\zeta(u,x)\zeta(v,x)=x^{-v}\zeta(u,x+1)+x^{-u}\zeta(v,x+1)+x^{-u-v},\quad u\neq 1,\ v\neq 1,$$

and

$$-\Delta\zeta(s,x)=x^{-s},\quad s\neq 1,$$

(which can be easily proved by direct calculation) we are naturally led to define $f_\kappa(x;u,v)$ by

$$\begin{aligned}f_\kappa(x;u,v)=\zeta(u,x)\zeta(v,x)-\zeta(u+v,x)&\\-\sum_{k=0}^{\kappa}\frac{\Gamma(u+k-1)}{\Gamma(u)k!}B_k\zeta(u+v+k-1,x)&\\-\sum_{k=0}^{\kappa}\frac{\Gamma(v+k-1)}{\Gamma(v)k!}B_k\zeta(u+v+k-1,x)&\end{aligned} \tag{2.7}$$

of which (1.4) is the special case with $\kappa=0$.

Now that f_κ is defined, we argue in the same way as in Section 1, and correspondingly to (1.5) and (1.6) we have

$$-\Delta f_\kappa(x;u,v)=g_\kappa(x;u,v)+g_\kappa(x;v,u), \tag{2.8}$$

$$g_\kappa(x;u,v)=x^{-v}\left(\zeta(u,x+1)-\sum_{k=0}^{\kappa}\frac{\Gamma(u+k-1)}{\Gamma(u)k!}B_k x^{-u-k+1}\right). \tag{2.9}$$

We shall see that κ, a temporary parameter taking large values, will disappear at the final stage. Integrating (2.9) by parts N-times, we have, correspondingly to (1.10),

$$\begin{aligned}
&\int_1^\infty g_\kappa(x;u,v)\,\mathrm{d}x\\
&\quad=-\sum_{n=0}^{N-1}\frac{(u)_n}{(1-v)_{n+1}}g_\kappa(1;u+n,v-n)+\frac{(u)_N}{(1-v)_N}\int_1^\infty g_\kappa(x;u+N,v-N)\,\mathrm{d}x\\
&\quad=-S_N(u,v)+S_2+S_3,
\end{aligned}\tag{2.10}$$

where

$$S_2=\sum_{n=0}^{N-1}\frac{1}{(1-v)_{n+1}}\sum_{k=0}^{\kappa}\frac{\Gamma(u+n+k-1)}{\Gamma(u)k!}B_k,$$

$$S_3=\frac{(u)_N}{(1-v)_N}\int_1^\infty x^{N-v}\left(\zeta(u+N,x+1)-\sum_{k=0}^{\kappa}\frac{\Gamma(u+N+k-1)}{\Gamma(u+N)k!}B_k x^{-u-N-k+1}\right)\mathrm{d}x.$$

To supplement the integral over $(1,\infty)$, we note that on the one hand, we have, for $\operatorname{Re}u+\operatorname{Re}v<2-\kappa$,

$$\int_0^1 x^{-v}\sum_{k=0}^{\kappa}\frac{\Gamma(u+k-1)}{\Gamma(u)k!}B_k x^{-u-k+1}\,\mathrm{d}x=-\sum_{k=0}^{\kappa}\frac{\Gamma(u+k-1)}{\Gamma(u)k!}B_k\frac{1}{u+v+k-2},$$

and on the other hand, by N-times integration by parts,

$$\begin{aligned}
&\int_0^1 x^{-v}\sum_{k=0}^{\kappa}\frac{\Gamma(u+k-1)}{\Gamma(u)k!}B_k x^{-u-k+1}\,\mathrm{d}x\\
&\quad=\sum_{n=0}^{N-1}\frac{1}{(1-v)_{n+1}}\sum_{k=0}^{\kappa}\frac{\Gamma(u+n+k-1)}{\Gamma(u)k!}B_k\\
&\qquad+\frac{(u)_N}{(1-v)_N}\int_0^1 x^{N-v}\sum_{k=0}^{\kappa}\frac{\Gamma(u+N+k-1)}{\Gamma(u+N)k!}B_k x^{-u-N-k+1}\,\mathrm{d}x,
\end{aligned}$$

whence, correspondingly to (1.14), we have

$$\frac{(u)_N}{(1-v)_N}\int_0^1 x^{N-v}\sum_{k=0}^{\kappa}\frac{\Gamma(u+N+k-1)}{\Gamma(u+N)k!}B_k x^{-u-N-k+1}\,\mathrm{d}x$$
$$=-\sum_{n=0}^{N-1}\frac{1}{(1-v)_{n+1}}\sum_{k=0}^{\kappa}\frac{\Gamma(u+n+k-1)}{\Gamma(u)k!}B_k - \sum_{k=0}^{\kappa}\frac{\Gamma(u+k-1)}{\Gamma(u)k!}B_k\frac{1}{u+v+k-2}. \tag{2.11}$$

And correspondingly to (1.12), we have, for $N-\operatorname{Re} v>1$, $\operatorname{Re} u+N>1$,

$$\frac{(u)_N}{(1-v)_N}\int_0^1 x^{N-v}\zeta(u+N,x+1)\,\mathrm{d}x = T_N(u,v). \tag{2.12}$$

For under the conditions on u and v above, the left-hand side of (2.12) can be written as

$$\sum_{l=1}^{\infty}\int_0^1 x^{N-v}(l+x)^{-u-N}\,\mathrm{d}x,$$

interchange of integration and summation being permissible by absolute convergence, and change of variables $\beta=\frac{l}{x}$ leads us to the integral in (1.13).

Thus, formula (2.10), after the integrals being supplemented, becomes

$$\int_1^{\infty} g_\kappa(x;u,v)\,\mathrm{d}x$$
$$=-S_N(u,v)-T_N(u,v) + \frac{(u)_N}{(1-v)_N}\int_0^{\infty} x^{N-v}\left(\zeta(u+N,x+1)-\sum_{k=0}^{\kappa}\frac{\Gamma(u+N+k-1)}{\Gamma(u+N)k!}B_k x^{-u-N-k+1}\right)\mathrm{d}x - \sum_{k=0}^{\kappa}\frac{\Gamma(u+k-1)}{\Gamma(u)k!}B_k\frac{1}{u+v+k-2}, \tag{2.13}$$

which corresponds to (1.15).

Substituting (2.13) and its counterpart $\int_1^{\infty} g_\kappa(x;u,v)\,\mathrm{d}x$ in

$$\int_1^2 f_\kappa(x;u,v)\,\mathrm{d}x = \int_1^{\infty} -\Delta f_\kappa(x;u,v)\,\mathrm{d}x = \int_1^{\infty} g_\kappa(x;u,v)\,\mathrm{d}x + \int_1^{\infty} g_\kappa(x;v,u)\,\mathrm{d}x, \tag{2.14}$$

the more general case than (1.9), we derive that

$$\begin{aligned}
&\int_1^2 f_\kappa(x;u,v)\,\mathrm{d}x \\
&\quad = -S_N(u,v) - S_N(v,u) - T_N(u,v) - T_N(v,u) \\
&\qquad + \frac{(u)_N}{(1-v)_N}\int_0^\infty x^{N-v}\left(\zeta(u+N,x+1) - \sum_{k=0}^{\kappa}\frac{\Gamma(u+N+k-1)}{\Gamma(u+N)k!}B_k x^{-u-N-k+1}\right)\mathrm{d}x \\
&\qquad + \frac{(v)_N}{(1-u)_N}\int_0^\infty x^{N-u}\left(\zeta(v+N,x+1) - \sum_{k=0}^{\kappa}\frac{\Gamma(v+N+k-1)}{\Gamma(v+N)k!}B_k x^{-v-N-k+1}\right)\mathrm{d}x \\
&\qquad - \sum_{k=0}^{\kappa}\frac{\Gamma(u+k-1)}{\Gamma(u)k!}B_k\frac{1}{u+v+k-2} - \sum_{k=0}^{\kappa}\frac{\Gamma(v+k-1)}{\Gamma(v)k!}B_k\frac{1}{u+v+k-2},
\end{aligned} \tag{2.15}$$

which is a general case of (1.16).

The final step of computing the Mellin transforms remains the same, using instead of (1.17) and (1.18),

$$\zeta(u+N,x+1) = \frac{1}{\Gamma(u+N)}\int_0^\infty \frac{t}{e^t-1}e^{-xt}t^{u+N-2}\,\mathrm{d}t, \tag{2.16}$$

$$\Gamma(u+N+k-1)x^{-u-N-k+1} = \int_0^\infty t^k e^{-xt}t^{u+N-2}\,\mathrm{d}t. \tag{2.17}$$

The result being the same as in Section 1, we infer for $1-\kappa < \operatorname{Re} u + \operatorname{Re} v < 2-k$ that

$$\begin{aligned}
&\int_1^2 f_\kappa(x;u,v)\,\mathrm{d}x \\
&\quad = -S_N(u,v) - S_N(v,u) - T_N(u,v) - T_N(v,u) \\
&\qquad + \Gamma(u+v-1)\zeta(u+v-1)\left(\frac{\Gamma(1-v)}{\Gamma(u)} + \frac{\Gamma(1-u)}{\Gamma(v)}\right) \\
&\qquad - \sum_{k=0}^{\kappa}\frac{\Gamma(u+k-1)}{\Gamma(u)k!}B_k\frac{1}{u+v+k-2} - \sum_{k=0}^{\kappa}\frac{\Gamma(v+k-1)}{\Gamma(v)k!}B_k\frac{1}{u+v+k-2}.
\end{aligned} \tag{2.18}$$

Using (1.7) and (2.7), we obtain the counterpart of (1.8) as follows:

$$\int_1^2 \zeta(u,x)\zeta(v,x)\,\mathrm{d}x = \int_1^2 f_\kappa(x;u,v)\,\mathrm{d}x + \frac{1}{u+v-1}$$

$$+\sum_{k=0}^{\kappa}\frac{\Gamma(u+k-1)}{\Gamma(u)k!}B_k\frac{1}{u+v+k-2}$$
$$+\sum_{k=0}^{\kappa}\frac{\Gamma(v+k-1)}{\Gamma(v)k!}B_k\frac{1}{u+v+k-2}. \tag{2.19}$$

Substituting (2.18) in (2.19) completes the proof of (1.21), which is fundamental in their theory of mean square of the Hurwitz zeta-function.

In the statement of the theorem, the restriction $1-\kappa < \mathrm{Re}\, u + \mathrm{Re}\, x < 2-\kappa$ ($\kappa \in \mathbb{N} \cup \{0\}$) is replaced by $\mathrm{Re}\, u + \mathrm{Re}\, v \notin \mathbb{Z}$. This is legitimate save for the case $\mathrm{Re}\, u + \mathrm{Re}\, v > 2$, and in this case, the above argument shall remain true by omitting all terms containing Bernoulli numbers.

There still remains the restriction that $2 \geqslant \mathrm{Re}\, u + \mathrm{Re}\, v \in \mathbb{Z}$. This case can be treated by similar reasoning, but we still give a more elaborate argument in Section 3 which clears such restrictions.

Katsurada [11] considered the slightly more general case of the Hurwitz–Lerch zeta-function $\sum_{n=0}^{\infty}\frac{e^{2\pi i n\xi}}{(n+x)^s}$, and deduced the generalization of their theorems including the theorem in this paper with the aid of the Mellin–Barnes integrals. There is, however, one black-box inserted in his proof, i.e. he substitutes the integral for the simple factor $\frac{1}{w-z}$, which is far from being natural. We may give a very natural and built-in structural proof of his results by considering

$$\psi(s,x,\xi)=\sum_{n=0}^{\infty}\frac{e^{2\pi i(n+x)\xi}}{(n+x)^s}.$$

Then the difference is

$$-\Delta\psi(s,x,\xi)=\frac{e^{2\pi i\xi x}}{x^s},$$

and the reasoning presented in Sections 1, 2 remains valid by modifying it by introducing the factor $e^{2\pi i n\xi}$ in appropriate places. We shall return to the proof of this and other types of problems (notably, power mean values of Dirichlet L-functions, cf., e.g., [12]) in the forthcoming papers.

3. A more elaborate proof

In the proof given in Section 2 one perceives some switch backs occurring, i.e. cancellation of terms after supplementing the integrals. This drawback can be avoided with the aid of the following new representation for the Hurwitz zeta-function in Lemma 1. Using this expression, the proof goes around the switch-back route, but becomes less direct.

Lemma 1. *Suppose $\alpha > 0$ and the complex variables s, z satisfy $\mathrm{Re}\, s > -M$ for $M \in \mathbb{N} \cup \{0\}$, $s \neq 1$ and $\mathrm{Re}(\alpha+z) > 0$. Then*

$$\zeta(s,\alpha+z)=\frac{1}{\Gamma(s)}\int_1^{\infty}\frac{e^{-(\alpha+z)t}}{1-e^{-t}}t^{s-1}\,\mathrm{d}t$$
$$+\frac{1}{\Gamma(s)}\int_0^1\left(\frac{te^{-zt}}{1-e^{-t}}-\sum_{m=0}^{M}\frac{B_m(z)}{m!}(-t)^m\right)e^{-\alpha t}t^{s-2}\,\mathrm{d}t$$

$$+\sum_{m=0}^{M}\frac{\Gamma(s+m-1)}{\Gamma(s)m!}(-1)^m B_m(z)\alpha^{-s-m+1}$$
$$-\sum_{m=0}^{M}\frac{\Gamma(s+m-1,\alpha)}{\Gamma(s)m!}(-1)^m B_m(z)\alpha^{-s-m+1}, \tag{3.1}$$

where $\Gamma(s,\alpha)$ *signifies the incomplete gamma function of the second kind defined for* $\mathrm{Re}\, s>0$*, by*

$$\Gamma(s,\alpha)=\int_{\alpha}^{\infty} e^{-t}t^{s-1}\,\mathrm{d}t. \tag{3.2}$$

Formula (3.1) *is still valid in the case* $\alpha=0$*,* $\mathrm{Re}\, z>0$ *in the following form*:

$$\zeta(s,z)=\frac{1}{\Gamma(s)}\int_{1}^{\infty}\frac{e^{-zt}}{1-e^{-t}}t^{s-1}\,\mathrm{d}t$$
$$+\frac{1}{\Gamma(s)}\int_{0}^{1}\left(\frac{te^{-zt}}{1-e^{-t}}-\sum_{m=0}^{M}\frac{B_m(z)}{m!}(-t)^m\right)t^{s-2}\,\mathrm{d}t$$
$$+\sum_{m=0}^{M}\frac{(-1)^m B_m(z)}{\Gamma(s)m!}\frac{1}{s+m-1}. \tag{3.3}$$

The integral $\int_1^\infty$ *in* (3.1) *converges uniformly in* s *for* $\mathrm{Re}(\alpha+z)>0$ *and the integral* $\int_0^1$ *converges absolutely for* $\sigma>-M$*.*

Proof. For $\mathrm{Re}\, s>1$, the integral representation for the Hurwitz zeta-function

$$\zeta(s,z+\alpha)=\frac{1}{\Gamma(s)}\int_{0}^{\infty}\frac{e^{-(\alpha+z)t}}{1-e^{-t}}t^{s-1}\,\mathrm{d}t \tag{3.4}$$

is well known [22]. Following Riemann's argument, we divide the interval $(0,\infty)$ into two parts $(0,1)$ and $[1,\infty)$, the latter of which is the first term of (3.1). Recalling that the factor $\frac{e^{-zt}}{1-e^{-t}}$ of the integrand is a generating function of the Bernoulli polynomial $B_m(z)$, we subtract $\sum_{m=0}^{M}\frac{B_m(z)}{m!}(-t)^m t^{s-2}$ from the integrand. Then we add the corresponding integral $\int_0^1$ which is

$$\sum_{m=0}^{M}\frac{B_m(z)}{m!}\alpha^{-s-m+1}(-1)^m\big(\Gamma(s+m-1)-\Gamma(s+m-1,\alpha)\big),$$

whence (3.1) follows.

In the case $\alpha = 0$, we argue in the same way, and the added integral is

$$\sum_{m=0}^{M} \frac{B_m(z)}{m!}(-1)^m \frac{1}{s+m-1},$$

which proves (3.3). □

Remark 1. Formulas (3.1) and (3.3) in the limit reduce to the corresponding formulas for the digamma function as $s \to 1$ after subtracting $\frac{1}{s-1}$ from both sides.

We shall apply the special case of (3.1) with $\alpha = x > 0$, $z = 1$, which we write down for completeness:

$$\begin{aligned}\zeta(s, x+1) = {} & \frac{1}{\Gamma(s)} \int_1^\infty \frac{e^{-xt}}{e^t - 1} t^{s-1}\,\mathrm{d}t \\ & + \frac{1}{\Gamma(s)} \int_0^1 \left(\frac{t}{e^t - 1} - \sum_{m=0}^{M} \frac{B_m}{m!} t^m \right) e^{-xt} t^{s-2}\,\mathrm{d}t \\ & + \sum_{m=0}^{M} \frac{\Gamma(s+m-1) B_m}{\Gamma(s) m!} x^{-s-m+1} \\ & - \sum_{m=0}^{M} \frac{\Gamma(s+m-1, x) B_m}{\Gamma(s) m!} x^{-s-m+1}. \end{aligned} \tag{3.5}$$

Lemma 2. *Denoting by $(a)_n$ the Pochhammer symbol, we have the evaluation*

$$\sum_{r=1}^{n} \frac{(b)_r}{(a)_r} = \frac{b}{b-a+1} \left(\frac{(b+1)_n}{(a)_n} - 1 \right). \tag{3.6}$$

Proof. For the hypergeometric function

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} x^n,$$

Whipple's relation

$$\begin{aligned} & (\alpha - \beta)(1 - x)\,{}_2F_1(\alpha, \beta; \gamma; x) + (\gamma - \alpha)\,{}_2F_1(\alpha - 1, \beta; \gamma; x) \\ & \quad + (\beta - \gamma)\,{}_2F_1(\alpha, \beta - 1; \gamma; x) = 0 \end{aligned}$$

is well known. Choosing $\alpha = b + 1$, $\beta = 1$, $\gamma = a$ and noting that ${}_2F_1(b+1, 0; a; x) = 1$, we deduce that

$$\frac{1}{1-x}\,{}_2F_1(b, 1; a; x) = \frac{b}{b-a+1}\,{}_2F_1(b+1, 1; a; x) - \frac{a-1}{(1-x)(b-a+1)}. \tag{3.7}$$

Comparing the Taylor coefficients of both sides of (3.7) completes the proof. □

Proof of the theorem. Now with $\kappa \geqslant 2N$, we define f_κ and g_κ as in (2.7) and (2.9), where in the case that $u+v$ is an integer $\leqslant 2$, we are to replace the term $\zeta(u+v+k-1, x)$ with $u+v+k-1=1$ by $-\psi(x)$ in (2.7). Then (1.8) holds, and (2.10) also follows.

To transform S_2 on the right of (2.10), we change the order of summation and apply Lemma 2 to the inner sum

$$\sum_{n=0}^{N-1} \frac{1}{(1-v)_{n+1}} \frac{\Gamma(u+n+k-1)}{\Gamma(u)} = \frac{\Gamma(u+k-2)}{\Gamma(u)} \sum_{n=0}^{N-1} \frac{(u+k-2)_{n+1}}{(1-v)_{n+1}}$$

to see that it is

$$\frac{\Gamma(u+k-2)}{\Gamma(u)} \frac{u+k-2}{u+v+k-2}\left(\frac{(u+k-1)_N}{(1-v)_N}-1\right).$$

Hence

$$\begin{aligned} S_2 = {} & \frac{1}{(1-v)_N} \sum_{k=0}^{\kappa} \frac{\Gamma(u+N+k-1)}{\Gamma(u)k!} \frac{B_k}{u+v+k-2} \\ & - \sum_{k=0}^{\kappa} \frac{\Gamma(u+k-1)}{\Gamma(u)k!} \frac{B_k}{u+v+k-2}. \end{aligned} \tag{3.8}$$

We now apply (3.5) to S_3 on the right of (2.10). Then

$$\begin{aligned} S_3 = {} & \frac{(u)_N}{(1-v)_N}\left[\frac{1}{\Gamma(u+N)} \int_1^\infty \int_1^\infty \frac{e^{-xt}}{e^t-1} t^{u+N-1} x^{N-v}\, \mathrm{d}t\, \mathrm{d}x \right. \\ & + \frac{1}{\Gamma(u+N)} \int_1^\infty \int_0^1 \left(\frac{t}{e^t-1} - \sum_{k=0}^{\kappa} \frac{B_k}{k!} t^k\right) e^{-xt} t^{u+N-2} x^{N-v}\, \mathrm{d}t\, \mathrm{d}x \\ & \left. - \sum_{k=0}^{\kappa} \frac{B_k}{\Gamma(u+N)k!} \int_1^\infty x^{-u-v-k+1} \Gamma(u+N+k-1, x)\, \mathrm{d}x \right] \\ = {} & \frac{(u)_N}{(1-v)_N}(S_{3,1}+S_{3,2}+S_{3,3}), \end{aligned} \tag{3.9}$$

say.

We transform $S_{3,1}$ and $S_{3,2}$ by completing the interval over $(1,\infty)$ in x to $(0,\infty)$, interchange the order of integration, and employing the integral

$$\int_0^\infty e^{-xt} x^{N-v}\, \mathrm{d}x = t^{v-N+1}\Gamma(1-v+N).$$

Thus

$$S_{3,1} = \frac{\Gamma(1-v+N)}{\Gamma(u+N)} \int_1^{\infty} \frac{t^{u+v-2}}{e^t-1}\,\mathrm{d}t - \frac{1}{\Gamma(u+N)} \int_0^1 \int_1^{\infty} \frac{e^{-xt}}{e^t-1} t^{u+N-1} x^{N-v}\,\mathrm{d}t\,\mathrm{d}x, \quad (3.10)$$

and

$$S_{3,2} = \frac{\Gamma(1-v+N)}{\Gamma(u+N)} \int_0^1 \left(\frac{t}{e^t-1} - \sum_{k=0}^{\kappa} \frac{B_k}{k!} t^k\right) t^{u+v-3}\,\mathrm{d}t$$
$$- \frac{1}{\Gamma(u+N)} \int_0^1 \int_0^1 \left(\frac{t}{e^t-1} - \sum_{k=0}^{\kappa} \frac{B_k}{k!} t^k\right) e^{-xt} t^{u+N-2} x^{N-v}\,\mathrm{d}t\,\mathrm{d}x. \quad (3.11)$$

To transform the integral $\int_1^{\infty} x^{-u-v-k+1}\Gamma(u+N+k-1,x)\,\mathrm{d}x$ in $S_{3,3}$ we substitute the integral expression for the incomplete gamma function and then invert the order of integration to get $\int_1^{\infty}\int_x^{\infty} \ldots \mathrm{d}t\,\mathrm{d}x = \int_1^{\infty}\int_1^{t} \ldots \mathrm{d}x\,\mathrm{d}t$. Then we supplement the integral over $(1,\infty)$ in t by adding $\int_0^1\int_t^1 \ldots \mathrm{d}x\,\mathrm{d}t$, whose order of integration we again invert as $\int_0^1\int_0^x \ldots \mathrm{d}t\,\mathrm{d}x$. The first integral $\int_0^{\infty}\int_1^{t} \ldots \mathrm{d}x\,\mathrm{d}t$ becomes

$$\frac{1}{u+v+k-2}\big(\Gamma(u+N+k-1) - \Gamma(1-v+N)\big),$$

while the second $\int_0^1\int_0^x \ldots \mathrm{d}t\,\mathrm{d}x$ becomes

$$\int_0^1 \big(\Gamma(u+N+k-1) - \Gamma(u+N+k-1,x)\big) x^{-u-v-k+1}\,\mathrm{d}x.$$

Hence

$$S_{3,3} = \frac{\Gamma(1-v+N)}{\Gamma(u+N)} \sum_{k=0}^{\kappa} \frac{B_k}{k!} \frac{1}{u+v+k-2} - \frac{1}{\Gamma(u+N)} \sum_{k=0}^{\kappa} \frac{B_k}{k!} \frac{\Gamma(u+N+k-1)}{u+v+k-2}$$
$$- \frac{1}{\Gamma(u+N)} \int_0^1 \sum_{k=0}^{\kappa} \big(\Gamma(u+N+k-1) - \Gamma(u+N+k-1,x)\big) \frac{B_k}{k!} x^{-u-v-k+1}\,\mathrm{d}x. \quad (3.12)$$

Substituting (3.10)–(3.12) in (3.9), we deduce that

$$S_3 = \frac{(u)_N}{(1-v)_N} \left(\frac{\Gamma(1-v+N)}{\Gamma(n+N)} \left\{ \int_1^{\infty} \frac{t^{u+v-2}}{e^t-1}\,\mathrm{d}t + \int_0^1 \left(\frac{t}{e^t-1} - \sum_{k=0}^{\kappa} \frac{B_k}{k!} t^k\right) t^{u+v-3}\,\mathrm{d}t\right.\right.$$

$$+\sum_{k=0}^{\kappa}\frac{B_k}{k!}\frac{1}{u+v+k-2}\Bigg\}-\frac{1}{\Gamma(u+N)}\int_0^1\Bigg[\int_1^\infty\frac{e^{-xt}}{e^t-1}t^{u+N-1}\,\mathrm{d}t$$

$$+\int_0^1\left(\frac{t}{e^t-1}-\sum_{k=0}^{\kappa}\frac{B_k}{k!}t^k\right)e^{-xt}t^{u+N-2}\,\mathrm{d}t$$

$$+\sum_{k=0}^{\kappa}\big(\Gamma(u+N+k-1)-\Gamma(u+N+k-1,x)\big)\frac{B_k}{k!}x^{-u-N-k+1}\Bigg]x^{N-v}\,\mathrm{d}x\Bigg)$$

$$-\frac{(u)_N}{(1-v)_N}\sum_{k=0}^{\kappa}\frac{\Gamma(u+N+k-1)}{\Gamma(u+N)k!}\frac{B_k}{u+v+k-2}.$$

Using (3.3) with $z=1$ and (3.5) for the first and the second integral, respectively, we conclude that

$$S_3=\frac{(u)_N}{(1-v)_N}\left\{\frac{\Gamma(1-v+N)}{\Gamma(u+N)}\Gamma(u+v-1)\zeta(u+v-1)-\int_0^1\zeta(u+N,x+1)x^{N-v}\,\mathrm{d}x\right\}$$
$$-\frac{(u)_N}{(1-v)_N}\sum_{k=0}^{\kappa}\frac{\Gamma(u+N+k-1)}{\Gamma(u+N)k!}\frac{B_k}{u+v+k-2},\tag{3.13}$$

or

$$S_3=\frac{\Gamma(u)}{\Gamma(1-v)}\Gamma(u+v-1)\zeta(u+v-1)-T_N(u,v)$$
$$-\frac{(u)_N}{(1-v)_N}\sum_{k=0}^{\kappa}\frac{\Gamma(u+N+k-1)}{\Gamma(u+N)k!}\frac{B_k}{u+v+k-2}.\tag{3.14}$$

Substituting (3.8) and (3.14) in (2.10) yields

$$\int_1^\infty g_\kappa(x;u,v)\,\mathrm{d}x=-S_N(u,v)-T_N(u,v)+\Gamma(u+v-1)\zeta(u+v-1)\frac{\Gamma(u)}{\Gamma(1-v)}$$
$$-\sum_{k=0}^{\kappa}\frac{\Gamma(u+k-1)}{\Gamma(u)k!}\frac{B_k}{u+v+k-2}.\tag{3.15}$$

Substituting (3.15) and its counterpart for $g_\kappa(x;u,v)$, we obtain

$$\int_1^2 f_\kappa(x;u,v)\,\mathrm{d}x$$
$$=-S_N(u,v)-S_N(v,u)-T_N(u,v)-T_N(v,u)$$

$$+\left(\frac{\Gamma(u)}{\Gamma(1-v)}+\frac{\Gamma(v)}{\Gamma(1-u)}\right)\Gamma(u+v-1)\zeta(u+v-1)$$
$$-\sum_{k=0}^{\kappa}\frac{\Gamma(u+k-1)}{\Gamma(u)k!}\frac{B_k}{u+v+k-2}-\sum_{k=0}^{\kappa}\frac{\Gamma(v+k-1)}{\Gamma(v)k!}\frac{B_k}{u+v+k-2}. \tag{3.16}$$

Now (2.19) and (3.16) completes the proof of the theorem in Section 1 in the case $2 \geqslant u + v \notin \mathbb{Z}$. In the case $2 \geqslant u + v \in \mathbb{Z}$, we must consider the term for which $u + v + k - 1 = 1$ separately as we remarked at the beginning of the proof. But in this case (2.19) and (3.16) hold in the form that the corresponding superfluous sums cancel each other, and the proof follows. □

4. Concluding remarks

As pointed out in Section 1, formula (1.8) reminds us of a formula of Wilton [24] (Wilton's formula is our formula (2.7) with $\kappa = 1$) and subsequent work of Bellman [5]. We remark here that a slight change of definition of g_0 leads to Atkinson's dissection formula quite naturally.

Define in analogy with (1.6),

$$\tilde{g}_0(x;u,v)=x^{-v}\zeta(u,x). \tag{4.1}$$

Then, comparing this with (1.6), we find that

$$g_0(x;u,v)=\tilde{g}_0(x;u,v)-\frac{1}{u-1}x^{1-u-v}-x^{-u-v}. \tag{4.2}$$

We apply Abel's summation in the following form. Suppose the sequences $\{a_n\}$, $\{b_n\}$ satisfy $\sum |a_n b_n| < \infty$ and, with $B(N)=\sum_{n=0}^{N} b_n$, $\lim_{N\to\infty} a_N B(N) = 0$. Then

$$\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty}(a_n-a_{n+1})B(n).$$

For $\operatorname{Re} u > 1$ and $\operatorname{Re} v > 1$ we choose $a_n = \zeta(u, x+n)$ and $b_n = (x+n)^{-v}$ to obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\tilde{g}_0(x+n;u,v)&=\sum_{n=0}^{\infty}(x+n)^{-v}\zeta(u,x+n)=\sum_{n=0}^{\infty}(x+n)^{-u}\sum_{m=0}^{n}(x+m)^{-v}\\&=\zeta_2(u,v;x)+\zeta(u+v,x),\end{aligned} \tag{4.3}$$

where

$$\zeta_2(u,v;x)=\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}\frac{1}{(m+x)^u}\frac{1}{(n+x)^v} \tag{4.4}$$

is one of the two Dirichlet series appearing in Atkinson's dissection and has been extensively studied by Matsumoto et al. (cf. [6]).

From (3.18) and (3.19) it follows that for $\operatorname{Re} u + \operatorname{Re} v > 2$

$$\sum_{n=0}^{\infty} g_0(x+n; u, v) = \sum_{n=0}^{\infty} \tilde{g}_0(x+n; u, v) - \frac{1}{u-1} \sum_{n=0}^{\infty} \frac{1}{(x+n)^{u+v-1}} - \sum_{n=0}^{\infty} \frac{1}{(x+n)^{u+v}}$$
$$= \zeta_2(u, v; x) - \frac{1}{u-1} \zeta(u+v-1, x), \tag{4.5}$$

whence that

$$\sum_{n=0}^{\infty} \bigl(-\Delta f_0(x+n; u, v)\bigr) = \zeta_2(u, v; x) + \zeta_2(v, u; x) - \zeta(u+v-1, x)\left(\frac{1}{u-1} + \frac{1}{v-1}\right). \tag{4.6}$$

Since (3.22) is $f_0(x; u, v)$ under the assumption of the convergence of the telescoping series, we conclude that Wilton's formula implies Atkinson's dissection formula.

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Further reading

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